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A Spectral Approach to Polynomial Matrices Solvents

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Abstract—Using the concept of Jordan chain of $L(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0$, where the A_i are $n \times n$ matrices with complex entries, we obtain a necessary and sufficient condition for the existence of $S \in \mathbb{C}^{n\times n}$ such that $L(S) = 0$. A way of finding those matrices, usually called solvents of $L(\lambda)$, is given as well.

Keywords—Matrix equations, Polynomial matrices, Solvents, Jordan form.

1. INTRODUCTION

Let $L(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0$ be a polynomial matrix, where $A_i \in \mathbb{C}^{n\times n}$, $i = 1, \dots, n$. We are concerned about the existence of $S \in \mathbb{C}^{n\times n}$ such that $A_lS^l + A_{l-1}S^{l-1} + \cdots + A_0 = 0$. Such an S is called a solvent of $L(\lambda)$. If solvents exist, we show how to construct them.

The direct approach to this problem is difficult. The issue is how to solve an $n^2 \times n^2$ nonlinear system. Most existent approaches are based on the monic case. Other resolutions for this problem can be found in [1–3]. However, as far as we know, there are no references to a necessary and sufficient condition for the existence of solvents for the nonmonic case. In this paper we follow the works of Gohberg, Rodman and Lancaster [4–6].

The main result is given by Theorem 1, which is a necessary and sufficient condition for the existence of solvents. It also gives a way of constructing those solvents, in case they exist.

Our starting point is the definition of a Jordan chain of generalized eigenvectors of $L(\lambda)$ (see [4]), as follows.

DEFINITION 1. *A sequence of column vectors $x_1, x_2, \dots, x_k$ in $\mathbb{C}^{n\times n}$ with x_1 not zero, satisfying the equalities*

$$\sum_{p=0}^{i-1} \frac{1}{p!} L^{(p)}(\lambda_0)x_{i-p} = 0, \qquad i = 1, \dots, k,$$

where $\lambda_0 \in \mathbb{C}$ and $L(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0$ and $L^{(p)}(\lambda)$ denotes the p^{th} derivative of $L(\lambda)$ with respect to λ, is called a Jordan chain of generalized eigenvectors of $L(\lambda)$ corresponding to λ_0, with length k.

This definition is a generalization of the usual notion of a Jordan chain of generalized eigenvectors of a square matrix with complex entries. As is well known, all sequence of vectors

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$x_1, x_2, \ldots, x_r$, with $x_1 \neq 0$, such that

$$\begin{aligned}(\lambda_0 I - A)x_1 &= 0\\ (\lambda_0 I - A)x_2 + x_1 &= 0\\ &\vdots\\ (\lambda_0 I - A)x_{r-1} + x_{r-2} &= 0\\ (\lambda_0 I - A)x_r + x_{r-1} &= 0,\end{aligned}$$

where $\lambda_0 \in \mathbb{C}$, is called a Jordan chain of A associated with the eigenvalue λ_0 with length r. The vectors forming that chain are usually called generalized eigenvectors. From these sequences, it is possible to obtain the Jordan Canonical Form for $A \in \mathbb{C}^{n\times n}$. Because of the similarity of that process with the one we are going to describe, we first present a review of the role played by Jordan chains when computing P and J such that

$$A = PJP^{-1},$$

where P is an invertible matrix and J a block-diagonal matrix.

Given $A \in \mathbb{C}^{n\times n}$ such that $\sigma(A) = \{\lambda_1, \ldots \lambda_s\}$, where $\sigma(A)$ is the spectrum of A, we can guarantee the existence of a Jordan chain set of A

$$B = \left(C_1^{(1)}, \ldots, C_{m_1}^{(1)}, \ldots, C_1^{(i)}, \ldots, C_{m_i}^{(i)}, \ldots, C_1^{(s)}, \ldots, C_{m_s}^{(s)}\right),$$

where $C_j^{(i)}$ represents a Jordan chain corresponding to λ_i, for $i = 1, \ldots, s$ and $j = 1, \ldots m_i$, such that B is formed by n linearly independent vector columns in $\mathbb{C}^{n\times n}$. Such a set is called a Jordan basis of A. Let

$$B_i = \left(C_1^{(i)}, \ldots, C_{m_i}^{(i)}\right)$$

and let us denote by $t_j^{(i)}$ the number of vector columns forming the Jordan chain $C_j^{(i)}$. Then

$$A = PJP^{-1}$$

with

$$P = [\, B_{J_1} \quad B_{J_2} \quad \cdots \quad B_{J_s} \,] \quad \text{and} \quad J = \begin{bmatrix} J_{\lambda_1} & & & 0 \\ & J_{\lambda_2} & & \\ & & \ddots & \\ 0 & & & J_{\lambda_s} \end{bmatrix}$$

for

$$J_{\lambda_i} = \begin{bmatrix} J_1^{(i)} & & & 0 \\ & J_2^{(i)} & & \\ & & \ddots & \\ 0 & & & J_{m_i}^{(i)} \end{bmatrix},$$

where every block is a Jordan block corresponding to λ_i; that is,

$$J_j^{(i)} = \begin{bmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ 0 & & & & \lambda_i \end{bmatrix}$$

is a Jordan block of order $t_j^{(i)}$ corresponding to λ_i, for $i = 1, \ldots, s$ and $j = 1, \ldots, m_i$. By Definition 1, such sets can be viewed as sets of Jordan chains of $L_1(\lambda) = \lambda I - A$. Notice also that $L_1(\lambda)$ has only one solvent which is A.

2. EXISTENCE OF SOLVENTS AND CONSTRUCTION

In the following, we consider the polynomial matrix

$$L(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0,$$

where coefficient matrices are of order n, and not necessarily nonsingular. We start with two lemmas where we introduce a necessary and sufficient condition for the existence of solvents. The first one can be found in [4; 5, p. 23; 6, p. 501] for the monic case. The second one appears in [5, p. 203] with a general formulation where the authors study the polynomial matrices' divisibility, and also, in [2, p. 21; 6] for monic matrices. We will present the proofs in a way fitted to our purposes.

LEMMA 1. *The vectors $x_1, \dots, x_k$ from $\mathbb{C}^n$, with $x_1 \neq 0$, form a Jordan chain of $L(\lambda)$ corresponding to λ_0 if and only if*

$$A_l X_0 J_0^l + A_{l-1} X_0 J_0^{l-1} + \cdots + A_0 X_0 = 0,$$

where $X_0 = [\,x_1 \quad \cdots \quad x_k\,]$ and J_0 is a Jordan block of order k with λ_0 on the main diagonal.

PROOF. First note that J_0^i, $i = 1, \dots, l$, is an upper triangular matrix of the order k. Each of its successive diagonals, starting from the main one, is formed by the same number. That is, throughout each diagonal we have the following scalars, corresponding to the indicated order:

$$\begin{aligned}
&\lambda_0^i, \binom{i}{1}\lambda_0^{i-1}, \dots, \binom{i}{i}\lambda_0^{i-i}, \underbrace{0, \dots, 0}_{k-(i+1)} \qquad && \text{if } i < k,\\
&\lambda_0^i, \binom{i}{1}\lambda_0^{i-1}, \dots, \binom{i}{i-k+1}\lambda_0^{i-k+1} && \text{if } i \geq k.
\end{aligned}$$

Then

$$A_0 X_0 + A_1 X_0 J_0 + \cdots + A_l X_0 J_0^l = [\,C_1 \quad C_2 \quad \cdots \quad C_k\,],$$

where

$$\begin{aligned}
C_1 &= L(\lambda_0)x_1,\\
C_2 &= L(\lambda_0)x_2 + L^{(1)}(\lambda_0)x_1,\\
&\ \vdots\\
C_k &= L(\lambda_0)x_k + \frac{1}{1!}L^{(1)}(\lambda_0)x_{k-1} + \cdots + \frac{1}{(k-1)!}L^{(k-1)}(\lambda_0)x_1.
\end{aligned}$$

So, $C_j = 0$, for $j = 1, 2, \dots, k$, if and only if $x_1, x_2, \dots, x_k$ form a Jordan chain of $L(\lambda)$. Therefore, $A_0X_0 + A_1X_0J_0 + \cdots + A_lX_0J_0^l = 0$ if and only if $x_1, x_2, \dots, x_k$ form a Jordan chain of $L(\lambda)$. ∎

The following lemma concerns the relationship between the set of Jordan chains of $L(\lambda)$ and the Jordan chains of its solvents.

LEMMA 2. *Let S be an $L(\lambda)$ solvent. Every Jordan chain of S corresponding to λ_0 is a Jordan chain of $L(\lambda)$ associated to λ_0.*

PROOF. Let S be a solvent of $L(\lambda)$ and $x_1, x_2, \dots, x_r$ a Jordan chain with length r corresponding to the eigenvalue λ_0 of S. We intend to prove that

$$\sum_{p=0}^{i-1} \frac{1}{p!} L^{(p)}(\lambda_0)x_{i-p} = 0, \qquad i = 1, \dots, r.$$

In other words, we must show that

$$L(\lambda_0)x_i + L^{(1)}(\lambda_0)x_{i-1} + \cdots + \frac{1}{(i-1)!}L^{(i-1)}(\lambda_0)x_1 = 0 \tag{1}$$

holds for every $i = 1, 2, \ldots, r$. Note that the left-hand side of (1) can be written in the following way:

$$\left[L(\lambda_0) + L^{(1)}(\lambda_0)(S - \lambda_0 I) + \cdots + \frac{1}{(i-1)!}L^{(i-1)}(\lambda_0)(S - \lambda_0 I)^{i-1}\right] x_i. \tag{2}$$

Since

$$L(\lambda) = Q(\lambda)(S - \lambda I)$$

where $Q(\lambda)$ is a polynomial matrix of order $l - 1$, we can state that

$$L^{(k)}(\lambda_0) = Q^{(k)}(\lambda_0)(S - \lambda_0 I) - kQ^{(k-1)}(\lambda_0), \qquad k \geq 1.$$

Therefore the coefficient of x_i in (2) is

$$\begin{array}{rl} 0 & \text{if } i > l, \\ \dfrac{1}{(i-1)!}Q^{(i-1)}(\lambda_0)(S - \lambda_0 I)^i & \text{if } i \leq l. \end{array}$$

Since x_i is a generalized eigenvector of S, it follows that coefficient of x_i in (2) is zero. ∎

The next theorem yields a necessary and sufficient condition for the solvent's existence. We emphasize the constructive nature of its proof.

THEOREM 1. *[7] A polynomial matrix $L(\lambda) = A_l\lambda^l + \cdots + A_1\lambda + A_0$, $n \times n$, has a solvent if and only if there is a set of Jordan chains of $L(\lambda)$ formed by n linear independent vectors.*

PROOF. Let S be solvent of $L(\lambda)$ and B_J one of its Jordan bases. The Jordan chains of S are, by Lemma 2, Jordan chains of $L(\lambda)$. So, by the properties of Jordan bases, B_J is a set of Jordan chains of $L(\lambda)$ formed by linearly independent vectors.

Conversely, let

$$\varphi_{11}, \ldots, \varphi_{1\mu_1};\ \varphi_{21}, \ldots, \varphi_{2\mu_2};\ \ldots;\ \varphi_{s1}, \ldots, \varphi_{s\mu_s}$$

be Jordan chains of $L(\lambda)$ corresponding, in the referred order, to the (not necessarily distinct) eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_s$$

where s is a natural number less than or equal to n, such that

$$\{\varphi_{11}, \ldots, \varphi_{1\mu_1}, \varphi_{21}, \ldots, \varphi_{2\mu_2}, \ldots, \varphi_{s1}, \ldots, \varphi_{s\mu_s}\}$$

is a linearly independent set formed by n elements from $\mathbb{C}^n$. If $X_1, X_2, \ldots, X_s$ are defined by

$$\begin{aligned} X_1 &= [\varphi_{11} \quad \cdots \quad \varphi_{1\mu_1}], \\ X_2 &= [\varphi_{21} \quad \cdots \quad \varphi_{2\mu_2}], \\ &\vdots \\ X_s &= [\varphi_{s1} \quad \cdots \quad \varphi_{s\mu_s}], \end{aligned}$$

then the matrix

$$P = [X_1 \quad X_2 \quad \cdots \quad X_s]$$

is an $n \times n$ nonsingular matrix. Let

$$J = \begin{bmatrix} J_{\lambda_1} & & & 0 \\ & J_{\lambda_2} & & \\ & & \ddots & \\ 0 & & & J_{\lambda_s} \end{bmatrix}$$

be a block-diagonal matrix where

$$J_{\lambda_i}, \qquad \text{for } i = 1, \dots, s$$

is a Jordan block of order μ_i with λ_i on the main diagonal. As we will prove, the matrix $S = PJP^{-1}$ is a solvent of $L(\lambda)$.

S is a solvent of $L(\lambda)$ if $L(S) = 0$, i.e.,

$$A_l P J^l P^{-1} + \cdots + A_1 P J P^{-1} + A_0 = 0,$$

that is, if

$$A_l P J^l + \cdots + A_1 P J + A_0 P = 0.$$

Considering the definitions for P and J, we get

$$\begin{aligned} A_l P J^l &= [\, A_l X_1 J_{\lambda_1}^l \quad A_l X_2 J_{\lambda_2}^l \quad \cdots \quad A_l X_s J_{\lambda_s}^l \,], \\ &\vdots \\ A_1 P J &= [\, A_1 X_1 J_{\lambda_1} \quad A_1 X_2 J_{\lambda_2} \quad \cdots \quad A_1 X_s J_{\lambda_s} \,], \\ A_0 P &= [\, A_0 X_1 \quad A_0 X_2 \quad \cdots \quad A_0 X_s \,]. \end{aligned}$$

Since by Lemma 1 we have

$$A_l X_i J_{\lambda_i}^l + \cdots + A_1 X_i J_{\lambda_i} + A_0 X_i = 0,$$

for $i = 1, \dots, s$, it follows that

$$A_l P J^l + \cdots + A_1 P J + A_0 P = 0,$$

and so S is a solvent of $L(\lambda)$. ∎

As a consequence of the proof of this theorem, we can obtain all solvents by knowing the whole set of Jordan chains of $L(\lambda)$. That strategy consists of obtaining all subsets B of Jordan chains, regarded as subsets of $\mathbb{C}^n$, which satisfy the following two conditions:

- B has size n;
- B linearly independent.

For each such set B, we can construct a solvent. According to the proof above, we take the vectors of B to be the columns of P, and taking J to be the corresponding Jordan matrix, we obtain the solvent $S = PJP^{-1}$ of $L(\lambda)$. If, on the other hand, it is not possible to form such a set B, then we can conclude that $L(\lambda)$ has no solvents.

3. EXAMPLES

We used the package MATLAB in the following computations.

EXAMPLE 1. Let

$$L(\lambda) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

This polynomial matrix has three eigenvalues: 0, with multiplicity 3; -1, with multiplicity 1; and 0.5, with multiplicity 1. The Jordan chains corresponding to 0 have maximum length 2 and are as follows:

$$\begin{array}{lll} (y,x,-y), & \text{for } x \neq 0 \quad \text{or} \quad y \neq 0 & \text{with length 1,} \\ (x,-x,-x);\ (z,t,x-z), & \text{for } x \neq 0, & \text{with length 2.} \end{array}$$

The ones corresponding to -1 and 0.5 have length 1 and are, respectively,

$$\begin{array}{ll} (u,-2u,0), & u \in \mathbb{C}\backslash\{0\}, \\ (w,-0.8w,0.6w), & w \in \mathbb{C}\backslash\{0\}. \end{array}$$

Using that information, we obtained several solvents, for example

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1.5 & 0.5 \\ -1 & -0.5 & -0.5 \end{bmatrix} \quad \begin{bmatrix} 1 & -0.5 & 1.5 \\ -1 & 0 & -1 \\ -1 & -0.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -2.5 \\ 2 & 0 & 4 \\ 0 & 0 & 0.5 \end{bmatrix} \quad \begin{bmatrix} -4 & -1.5 & -5.5 \\ 6 & 2 & 8 \\ 1 & 0.5 & 1.5 \end{bmatrix} \quad \begin{bmatrix} -19 & -9 & -20.5 \\ 26 & 12 & 28 \\ 6 & 3 & 6.5 \end{bmatrix}$$

and

$$\begin{bmatrix} -1.057 & -0.0285 & -2.557 \\ 2.076 & 0.038 & 4.076 \\ 0.019 & 0.0095 & 0.519 \end{bmatrix} \quad \begin{bmatrix} -1.03 & -0.015 & -2.53 \\ 2.04 & 0.02 & 4.04 \\ 0.01 & 0.005 & 0.51 \end{bmatrix}.$$

EXAMPLE 2. Let

$$L(\lambda) = A_2\lambda^2 + A_1\lambda + A_0,$$

where

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

In this case, $\sigma(L(\lambda)) = \{1,-1,0\}$. For the eigenvalue 1, the Jordan chains do not exceed the length 1 and are

$$(x,4x,-4x,2x), \qquad x \in \mathbb{C}\backslash\{0\}.$$

For the eigenvalue -1, the Jordan chains with length 1 are

$$(y,0,0,0), \qquad y \in \mathbb{C}\backslash\{0\},$$

and of length 2

$$(y,0,0,0),(0,0,0,y), \qquad y \in \mathbb{C}\backslash\{0\}.$$

Finally, for 0, the Jordan chains could be of length 1, 2, or 3:

$$\begin{array}{rll} (w,0,0,0), & w \in \mathbb{C}\backslash\{0\}, & \\ (t,0,0,0),(z,t,-t,0), & t \in \mathbb{C}\backslash\{0\}, & z \in \mathbb{C}, \\ (u,0,0,0),(v,u,-u,0),(s,u+v,-u-v,-u), & u \in \mathbb{C}\backslash\{0\}, & v,s \in \mathbb{C}. \end{array}$$

An analysis shows that these chains cannot be put together to form a set of n linearly independent vectors. So, we conclude that $L(\lambda)$ has no solvents in this case.

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